Pavani, G.A.
Osaka J. Math.
56 (2019), 577–599

ON A CLASS OF RAUZY FRACTALS WITHOUT THE FINITENESS PROPERTY

Gustavo A. PAVANI

(Received November 25, 2016, revised March 19, 2018)

Abstract

We present some topological and arithmetical aspects of a class of Rauzy fractals $\mathcal{R}_{a,b}$ related to the polynomials of the form $P_{a,b}(x) = x^3 - ax^2 - bx - 1$, where a and b are integers satisfying $-a + 1 \leq b \leq -2$. This class has the property that 0 lies on the boundary of $\mathcal{R}_{a,b}$. We construct explicit finite automata that recognize the boundaries of these fractals. This allows to establish the number of neighbors of $\mathcal{R}_{a,b}$ in the tiling it generates. Furthermore, we prove that if $2a + 3b + 4 \leq 0$ then $\mathcal{R}_{a,b}$ is not homeomorphic to a topological disk. We also show that the boundary of the set $\mathcal{R}_{3,-2}$ is generated by two infinite iterated function systems.

1. Introduction

Our aim is to study a class of Rauzy fractals related to algebraic integers β which do not satisfy a certain property called Property (F). This study involves fractal tilings, automata, $\beta-$numeration, and infinite iterated function systems (IIFS).

The Rauzy fractal was introduced by G. Rauzy in 1982 [27] and it is the set

$$\mathcal{E} = \left\{ \sum_{i=0}^{+\infty} \ell_i \alpha^i,\ \ell_i \in \{0, 1\},\ \ell_i \ell_{i+1} \ell_{i+2} = 0,\ \forall i \geq 0 \right\},$$

where α is one of the complex roots of the polynomial $x^3 - x^2 - x - 1$.

There are several ways to construct the Rauzy fractal, one of them is by using substitutions. Let $\mathcal{A} = \{1, 2, ..., d\}$, $d \geq 2$, be a finite alphabet. A word is a finite string of elements in $\mathcal{A}$. The set of all finite words over $\mathcal{A}$ is denoted by $\mathcal{A}^*$, and the empty word is denoted by ε. A substitution σ is an application from the alphabet $\mathcal{A}$ onto the set $\mathcal{A}^* \backslash \{\varepsilon\}$ of nonempty finite words on $\mathcal{A}$, and it extends to a morphism of $\mathcal{A}^*$ by concatenation, i.e., $\sigma(ww') = \sigma(w)\sigma(w')$. The substitution σ can be naturally extended to the set of infinite words $\mathcal{A}^{\mathbb{N}}$. The initial aim of Rauzy was to establish a geometric representation to the symbolic dynamical system associated with the substitution σ given by $\sigma(0) = 01$, $\sigma(1) = 02$, $\sigma(2) = 0$. Since then, this set and its generalizations have been studied by many mathematicians, due to the strong connections with other fields of mathematics, for instance, tilings [26, 1, 4], numeration systems [22, 23, 26], Markov partitions for toral automorphisms [21, 26, 17], geometric representation of symbolic dynamical systems [10, 4, 5, 6, 21, 32, 14, 31], simultaneous diophantine approximations [3, 11, 15], and the theory of quasicrystals [2].

The Rauzy fractal has remarkable properties (see [27]): it is a compact and connected subset of $\mathbb{C}$, its interior is simply connected, and it induces a periodic tiling of the complex

2010 Mathematics Subject Classification. Primary 11B85; Secondary 28A80, 37B10, 52C20.

plane. Moreover, it is divided into three self-similar copies of itself which correspond to an exchange of domains.

Another way to obtain the Rauzy fractal is via β−representation. Given a real number $\beta > 1$, a β-representation (or β-expansion) of a number $x \in \mathbb{R}^+$ is an infinite sequence $(x_i)_{i\leq k}$, where $k \in \mathbb{Z}$, $x_i \geq 0$ such that $x = \sum_{i=-\infty}^{k} x_i\beta^i$. The digits x_i can be computed using the greedy algorithm as follows (see [24, 12] for details): denote by $\lfloor x \rfloor$ and $\{x\}$ the integer and fractional parts of the number x. There exists $k \in \mathbb{Z}$ such that $\beta^k \leq x < \beta^{k+1}$. Let $x_k = \left\lfloor x/\beta^k \right\rfloor$ and $q_k = \left\{x/\beta^k\right\}$. Then, for $i < k$, put $x_i = \lfloor \beta q_{i+1} \rfloor$ and $q_i = \{\beta q_{i+1}\}$. We obtain $x = x_k\beta^k + x_{k-1}\beta^{k-1} + \cdots$. If $k < 0$ ($x < 1$) we put $x_0 = x_1 = \cdots = x_{k+1} = 0$. If a β-representation ends up with infinitely many zeros, it is said to be finite and the ending zeros can be omitted. Then, the sequence will be denoted by $(x_i)_{n\leq i\leq k}$ or $x_k \cdots x_n$. The digits x_i belong to the set $B = \{0, \cdots, \beta\}$, if β is an integer, and to the set $B = \{0, \cdots, \lfloor \beta \rfloor\}$, otherwise.

In particular, when β is a Pisot number, i.e., an algebraic integer greater then 1 such that all its Galois conjugates have modulus less than 1, we obtain classes of Rauzy fractals associated to these Pisot numbers. Cubic Pisot units were classified by Akiyama in [1] as being exactly the set of dominant roots of the polynomial $P_{a,b}(x) = x^3 - ax^2 - bx - 1$, satisfying one of the following conditions

a) $1 \leq b \leq a$ and $d(1, \beta) = .ab1$;

b) $b = -1$, $a \geq 2$ and $d(1, \beta) = .(a-1)(a-1)01$;

c) $b = a + 1$ and $d(1, \beta) = .(a + 1)00a1$;

d) $-a + 1 \leq b \leq -2$ and $d(1, \beta) = .(a-1)(a + b - 1)(a + b)^\infty$,

where $(a + b)^\infty$ is the periodic expansion $(a + b)(a + b)(a + b)\ldots$, and $d(1, \beta)$ is the Rényi β-representation of 1 (see [28] for the definition).

Let $\mathrm{Fin}(\beta)$ be the set of nonnegative real numbers that have a finite β-representation. We say that a Pisot number β has the Finiteness Property (or Property (F)) if $\mathbb{Z}[\beta] \cap [0, +\infty[\subset \mathrm{Fin}(\beta)$. Therefore, the Pisot numbers in the sets **a)**, **b)** and **c)** have the Property (F), while the Pisot numbers in **d)** have not. Many works were done for the classes **a)** and **b)** (see [35, 29, 22, 20, 7, 19]).

In this paper we will study the properties of the Rauzy fractals associated to the class of Pisot numbers which do not satisfy the Property (F), that is, the case where $-a+1 \leq b \leq -2$. As we shall see, this class shares common features with the others previously studied. For instance, these fractals sets are compact and they tile the plane. In fact, this class of fractals can be obtained via β−substitution defined by $\sigma(1) = 1^{(a-1)}2$, $\sigma(2) = 1^{(a+b-1)}3$, $\sigma(3) = 1^{(a+b)}3$ (see [8]). This substitution belongs to the class of the so-called Pisot substitutions and some topological properties for fractal sets arising from Pisot substitutions are well known (see [9, 10]). On the other hand, zero is not an inner point for the fractals of this class, contrary to the fractals associated with Pisot numbers that satisfy Property (F). In this work, we obtain explicit finite state automata that generate the boundary of $\mathcal{R}_{a,b}$. These automata lead to several results: we obtain a formula for the number of the neighbors of $\mathcal{R}_{a,b}$ in the periodic tiling and we prove that if $2a + 3b + 4 \leq 0$, then $\mathcal{R}_{a,b}$ is not homeomorphic to a topological disk. We study in more details the boundary of the set $\mathcal{R}_{3,-2}$, in particular we prove that the boundary of $\mathcal{R}_{3,-2}$ is generated by two infinite iterated function systems.

Notice that the set $\mathcal{R}_{3,-2}$ is related to a Special Pisot number, i.e., a Pisot number β such that $\beta/(\beta-1)$ is also a Pisot number (see [18, 33]).

This paper is organized in this way. In Section 2 we briefly describe the $\beta-$numeration background necessary to define the Rauzy fractal sets that we are considering. In Section 3 we give some properties of the boundary of $\mathcal{R}_{a,b}$. In Section 4 we construct the automata that recognize the boundaries of the sets $\mathcal{R}_{a,b}$. Finally, in Section 5, we use an automaton to obtain two infinite iterated function system for the boundary of $\mathcal{R}_{3,-2}$, and we show a geometric method in order to parametrize the boundary of $\mathcal{R}_{3,-2}$. In Section 6 (Annex) we show algebraically that $\mathcal{R}_{3,-2}$ has exactly six neighbors.

2. Numeration system and Rauzy fractal

In the sequel, we will suppose that β is a cubic Pisot unit which does not satisfy the Property (F) and we will denote by α and λ its Galois conjugates. Let $P_{a,b}(x) = x^3 - ax^2 - bx - 1$ be the minimal polynomial of β. Next, we will consider a generalization of numeration system induced by the β-expansion which only can be applied on integer numbers.

Let $(T_n)_{n\geq 0}$ be the recurrent sequence defined by $T_0 = 1$, $T_1 = a$, $T_2 = a^2 + b$, $T_{n+3} = aT_{n+2} + bT_{n+1} + T_n$, satisfying the condition $-a + 1 \leq b \leq -2$ for all $n \geq 0$.

Proposition 2.1. *Every nonnegative integer n can be uniquely expressed as $n = \sum_{i=0}^{N} \ell_i T_i$, where $\ell_i \in \{0, \dots, a-1\}$ and $\ell_j \ell_{j-1} \cdots \ell_{j-k} \leq_{lex} (a-1)(a+b-1)(a+b)\cdots(a+b)$, for all $j \geq k \geq 0$, where "$\leq_{lex}$" is the lexicographical order.*

For the proof we need the following lemma.

Lemma 2.2. *The sequence $(T_n)_{n\geq 4}$ satisfies*

$$T_n = (a-1)T_{n-1} + (a+b-1)T_{n-2} + (a+b)T_{n-3} + \cdots + (a+b)T_1 + (a+b+1)T_0,$$

for all $n \geq 4$.

Proof. The proof is by recurrence on n. It is not difficult to verify that the relation is valid for $n = 4, 5, 6$. Let $n \geq 7$ and suppose that the relation holds for all $k < n$. Since $T_n = aT_{n-1} + bT_{n-2} + T_{n-3}$, then $T_n = aT_{n-1} + bT_{n-2} + Q$, where

$$Q = (a-1)T_{n-4} + (a+b-1)T_{n-5} + (a+b)T_{n-6} + \cdots + (a+b)T_1 + (a+b+1)T_0.$$

Then,

$$\begin{aligned} aT_{n-1} + bT_{n-2} + (a-1)T_{n-4} + (a+b-1)T_{n-5} = (a-1)T_{n-1} &+ (a+b-1)T_{n-2} \\ &+ (a+b)T_{n-3} + (a+b)T_{n-4} \\ &+ (a+b)T_{n-5}. \end{aligned}$$

In fact,

$$\begin{aligned} &aT_{n-1} + bT_{n-2} + (a-1)T_{n-4} + (a+b-1)T_{n-5} \\ &\quad = (a-1)T_{n-1} + (a+b)T_{n-2} + bT_{n-3} + aT_{n-4} + (a+b-1)T_{n-5} \\ &\quad = (a-1)T_{n-1} + (a+b-1)T_{n-2} + (a+b)T_{n-3} + (a+b)T_{n-4} + (a+b)T_{n-5}. \qquad \square \end{aligned}$$

Proof of Proposition 2.1. Let $N \geq 0$ and $(\ell_j)_{0\leq j\leq N}$ be obtained by using the greedy algorithm. Since $-a + 1 \leq b \leq -2$, then $(T_n)_{n\geq 0}$ is an increasing sequence of natural integers. Hence, by the definition of the greedy algorithm, we can prove that

$$\sum_{i=0}^{j} \ell_i T_i < T_{j+1},$$

for all $0 \leq j \leq N$ (see [24]). Thus,

$$\ell_j \ell_{j-1} \cdots \ell_{j-k} <_{lex} (a-1)(a+b-1)(a+b)\cdots(a+b)(a+b+1), \ \forall j \geq k \geq 0.$$

Therefore, we obtain that $\ell_j \cdots \ell_{j-k} \leq_{lex} (a-1)(a+b-1)(a+b)\cdots(a+b)$. □

Let $\mathcal{L} = \{(\ell_i)_{i\geq k}, k \in \mathbb{Z}, \forall n \geq k, \ell_n \cdots \ell_{n-k} \leq_{lex} (a-1)(a+b-1)(a+b)\cdots(a+b)\}$. Then, the Rauzy fractal is the set

$$\mathcal{R} := \mathcal{R}_{a,b} = \left\{ \sum_{i=2}^{+\infty} \ell_i \theta_i, \ (\ell_n)_{n\in\mathbb{Z}} \in \mathcal{L} \right\}$$

where $\theta_i = \alpha^i$, if $\alpha \in \mathbb{C} \setminus \mathbb{R}$ or $\theta_i = (\alpha^i, \lambda^i)$, if $\alpha \in \mathbb{R}$. Observe that $\mathcal{R} \subset \mathbb{C}$ or $\mathcal{R} \subset \mathbb{R}^2$. Notice also that $\mathcal{R}$ is a compact set.

Example 2.3. 1. If $a = 3$ and $b = -2$, we can show that $P_{3,-2}(x) = x^3 - 3x^2 + 2x - 1$ has one real root $\beta > 1$ and two complex conjugates roots $\alpha, \overline{\alpha}$ wich satisfy $|\alpha|, |\overline{\alpha}| < 1$. In this case $\alpha \approx 0.33764 + 0.56228i$. The Rauzy fractal (Figure 1) is

$$\mathcal{R}_{3,-2} = \{\textstyle\sum_{i=2}^{+\infty} \ell_i \alpha^i, \forall j \geq n \geq 2, \ell_j \ell_{j-1} \cdots \ell_n \leq_{lex} 201 \cdots 1\}.$$

2. If $a = 6$ and $b = -5$, then $P_{6,-5}(x) = x^3 - 6x^2 + 5x - 1$ has three real roots: $\beta \approx 5.048917340$, $\alpha \approx 0.3079785280$ and $\lambda \approx 0.6431041320$. The Rauzy fractal (Figure 2) in this case is

$$\mathcal{R}_{6,-5} = \{(\textstyle\sum_{i=2}^{+\infty} \ell_i \alpha^i, \sum_{i=2}^{+\infty} \ell_i \lambda^i), \forall j \geq n \geq 2, \ell_j \ell_{j-1} \cdots \ell_n \leq_{lex} 501 \cdots 1\}.$$

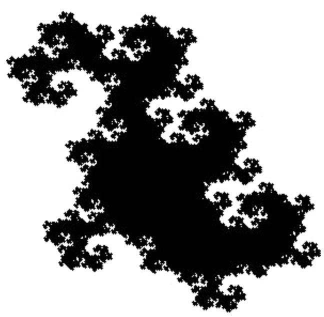

Fig. 1. The set $\mathcal{R}_{3,-2}$.

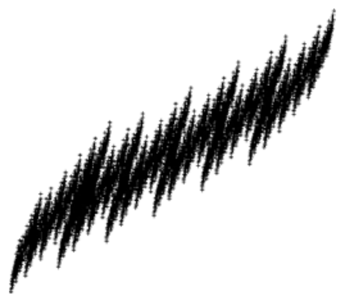

Fig. 2. The set $\mathcal{R}_{6,-5}$.

Remark 2.4. There are several ways to construct Rauzy fractals, one of them is via substitutions, as mentioned in the Introduction. Our class of Rauzy fractals is obtained by the substitution over a three-letter alphabet $\mathcal{A} = \{1, 2, 3\}$ given by $\sigma(1) = 1^{(a-1)}2$, $\sigma(2) = 1^{(a+b-1)}3$, $\sigma(3) = 1^{(a+b)}3$, provided that $-a + 1 \leq b \leq -2$. There is a personal website (see [16]), where one can draw online Rauzy fractals associated with any substitution, in particular to that one of our case. For more details on how to construct Rauzy fractals using substitutions, the reader is referred to [31].

3. Boundary of $\mathcal{R}$

In Sections 3 and 4, we will suppose that $\alpha \in \mathbb{C} \setminus \mathbb{R}$, but all the results that we will state remain valid for the case when $\alpha \in \mathbb{R}$. In this section we show some properties concerning the boundary of the Rauzy fractals. We will denote the interior of the set $\mathcal{R}$ by $int(\mathcal{R})$. We have the following theorem (see Fig. 3 and 4).

Theorem 3.1. *The Rauzy fractal $\mathcal{R} = \mathcal{R}_{a,b}$ induces a periodic tiling of the complex plane by the group $G = \mathbb{Z} + \mathbb{Z}\alpha$, that is, $\mathbb{C} = \bigcup_{u\in G}(\mathcal{R}+u)$ and $int(\mathcal{R}+u) \cap (\mathcal{R}+v) \neq \emptyset$ implies that $u = v$.*

Proof. The proof can be deduced from the work of Rauzy [27] or from the work of Canterini and Siegel [10], where the authors explicitly deal with Pisot substitutions. □

REMARK 3.2. We take the summation beginning from 2 in the definition of the Rauzy fractal to have the tiling group $\mathbb{Z} + \mathbb{Z}\alpha$. If we begin at 0, then the group is $\mathbb{Z}\alpha^{-2} + \mathbb{Z}\alpha^{-1}$.

Proposition 3.3. *The boundary $\partial\mathcal{R}$ of $\mathcal{R}$ satisfies the property:*
$\partial\mathcal{R} = \bigcup_{u\in H}\mathcal{R} \cap (\mathcal{R}+u)$, where H is a finite subset of $G = \mathbb{Z} + \mathbb{Z}\alpha$ whose cardinality is even and greater then or equal to 6. Moreover, $\{\pm(1 + (b + 1)\alpha), \pm\alpha, \pm(1 + b\alpha)\} \subset H$.

For the proof we need the following result (see Fig. 7).

Lemma 3.4. *Let $\psi : \{0, 1, \cdots, a - 1\}^{\mathbb{N}} \to \mathbb{C}$ defined by $\psi(\ell_0\ell_1\cdots) = \sum_{i=0}^{\infty} \ell_i\alpha^i$. Let*

$$w_1 = \psi((0000(b + 2)(a + b)(a - 2))^{\infty}),$$
$$w_2 = \psi(01b(a - 1)(000(b + 2)(a + b)(a - 2))^{\infty}),$$
$$w_3 = \psi(1(b + 1)(a + b)(a - 2)(000(b + 2)(a + b)(a - 2))^{\infty}),$$
$$z_1 = \psi(1b(a - 1)),\ z_2 = \psi(000(b + 2)(a + b + 1)(a + b)^{\infty}).$$

Then $w_1 = w_2 = w_3$, $z_1 = z_2$, and hence $w_1 \in \mathcal{R} \cap (\mathcal{R} + \alpha) \cap (\mathcal{R} + 1 + (b + 1)\alpha)$, and also $z_1 \in \mathcal{R} \cap (\mathcal{R} + 1 + b\alpha)$.

Proof. Let us show that $w_1 = w_2$. We have,

$$w_1 = \tfrac{1}{1-\alpha^6}((b + 2)\alpha^4 + (a + b)\alpha^5 + (a - 2)\alpha^6)$$

and

$$w_2 = \alpha + b\alpha^2 + (a - 1)\alpha^3 + \tfrac{1}{1-\alpha^6}((b + 2)\alpha^7 + (a + b)\alpha^8 + (a - 2)\alpha^9).$$

Then,

$$\begin{aligned} w_1 - w_2 = 0 &\iff (b + 2)\tfrac{(\alpha^4-\alpha^7)}{1-\alpha^6} + (a + b)\tfrac{(\alpha^5-\alpha^8)}{1-\alpha^6} + (a - 2)\tfrac{(\alpha^6-\alpha^9)}{1-\alpha^6} - \alpha - b\alpha^2 - (a - 1)\alpha^3 = 0 \\ &\iff \alpha^4(b + 2)\tfrac{(1-\alpha^3)}{1-\alpha^6} + \alpha^5(a + b)\tfrac{(1-\alpha^3)}{1-\alpha^6} + \alpha^6(a - 2)\tfrac{(1-\alpha^3)}{1-\alpha^6} \\ &\qquad - \alpha - b\alpha^2 - (a - 1)\alpha^3 = 0 \end{aligned}$$

Multiplying the last equation by $1 + \alpha^3$ we obtain

$$\alpha^4(b + 2) + \alpha^5(a + b) + \alpha^6(a - 2) - \alpha(1 + \alpha^3) - b\alpha^2(1 + \alpha^3) - \alpha^3(a - 1)(1 + \alpha^3) = 0.$$

Now, working out the left side of the above equation and using the fact that $\alpha^3 = a\alpha^2 + b\alpha + 1$ we obtain that $w_1 = w_2$.

The other cases can be done in the same way. □

Proof of Proposition 3.3. Notice that the set of $G-$translates intersecting $\mathcal{R}$ is finite since G is a lattice and $\mathcal{R}$ is compact. Hence the set H is finite and its cardinality is even because, if $u \in H$, then $-u \in H$. Let us prove that $\{\pm(1+(b+1)\alpha, \pm\alpha, \pm(1+b\alpha)\} \subset H$. By Lemma 3.4, we have seen that $w_1 \in \mathcal{R} \cap (\mathcal{R}+\alpha) \cap (\mathcal{R}+1+(b+1)\alpha)$. Therefore $-\alpha$ and $-1-(b+1)\alpha$ belong to H. We have also seen that $z_2 \in \mathcal{R} \cap (\mathcal{R}+1+b\alpha)$. Therefore, $-1-b\alpha$ belongs to H. □

REMARK 3.5. We have seen in Lemma 3.4 that a point, for instance w_1, belongs to $\mathcal{R} \cap (\mathcal{R}+\alpha)$. This means that w_1 has two ways to be represented. Actually, in that case, w_1 could be expressed in three different ways. Points like w_1 are said to have at least two α-representations. These points will be characterized in the next section.

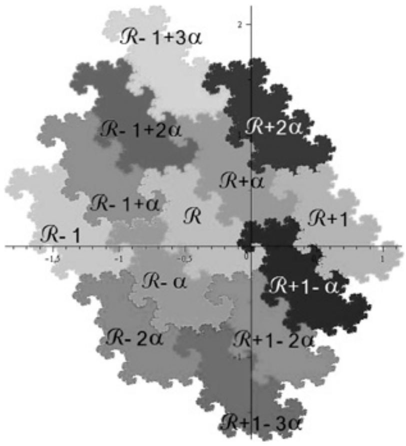

Fig. 3. Tiling the plane by $\mathcal{R}_{4,-3}$.

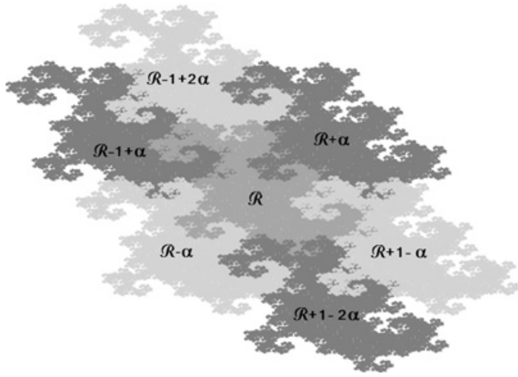

Fig. 4. $\mathcal{R}_{3,-2}$ and its 6 neighbors.

4. Construction of the automaton $\mathcal{G}$

In this section we prove that there exists an explicit and finite automaton that recognizes the points with two representations. These points belong to the boundary of $\mathcal{R}$. Let us begin with the following result.

Proposition 4.1. *Let $x = \sum_{i=l}^{\infty} a_i\alpha^i$ and $y = \sum_{i=l}^{\infty} b_i\alpha^i$, where $l \in \mathbb{Z}$ and $(a_i)_{i\geq l}$, $(b_i)_{i\geq l}$ belong to $\mathcal{L}$. Then $x = y$ if, and only if, the set $J(x,y) = \{x(k) - y(k), k \geq l\}$ is finite, where $x(k) = \alpha^{-k+2}\sum_{i=l}^{k} a_i\alpha^i$ and $y(k) = \alpha^{-k+2}\sum_{i=l}^{k} b_i\alpha^i$, $\forall k \geq l$.*

Moreover, $\bigcup_{(x,y)} J(x,y) \supset E_{a,b} = \{0, \pm\alpha^2, \pm(\alpha+b\alpha^2), \pm(\alpha+(b+1)\alpha^2), \pm(1+b\alpha+(a-1)\alpha^2), \pm(1+(b+1)\alpha+(a+b)\alpha^2), \pm(1+(b+1)\alpha+(a+b+1)\alpha^2)\}$.

Before proving the proposition, we will construct the automaton.

4.1. Algorithmic construction of the complex numbers that have two representations. Let p and q be two states. The set of edges is the set of $(p,(c,d),q) \in E_{a,b} \times \{0,1,...,a-1\}^2 \times E_{a,b}$ satisfying $q = \frac{p}{\alpha} + (c-d)\alpha^2$. The set of initial states is $\{0\}$.

Let us explain how this automaton acts. Let $x = \sum_{i=l}^{+\infty} a_i\alpha^i$ and $y = \sum_{i=l}^{+\infty} b_i\alpha^i$, where

$a = (a_i)_{i\geq l}$ and $b = (b_i)_{i\geq l}$ belong to $\mathcal{L}$. Suppose that $x = y$ and for all $k \geq l$ we set $S_k = S_k(a, b) = x(k) - y(k)$. We have,

$$S_{k+1} = \frac{S_k}{\alpha} + (a_{k+1} - b_{k+1})\alpha^2. \tag{1}$$

Let t be the smallest integer such that $a_t \neq b_t$. Hence $S_i(a, b) = 0$ for all $i \in \{l, ..., t-1\}$. Suppose that $a_t > b_t$. Then, $S_t = (a_t - b_t)\alpha^2 = \alpha^2$. From (1) we deduce that $S_{t+1} = \alpha + (a_{t+1} - b_{t+1})\alpha^2$ which should belong to $E_{a,b}$. Hence $S_{t+1} = \alpha + b\alpha^2$ if $(a_{t+1}, b_{t+1}) = (s_1 + b, s_1)$, where $0 \leq s_1 \leq a-1$, or $S_{t+1} = \alpha + (b+1)\alpha^2$, if $(a_{t+1}, b_{t+1}) = (t_1 + b + 1, t_1)$, where $0 \leq t_1 \leq a-1$. Continuing with this process, we obtain an infinite path $(S_i, (a_i, b_i), S_{i+1})_{i\geq l}$ beginning in the initial state of the finite state automaton (see Fig. 5). This path will be denoted by $(a_i, b_i)_{i\geq l}$.

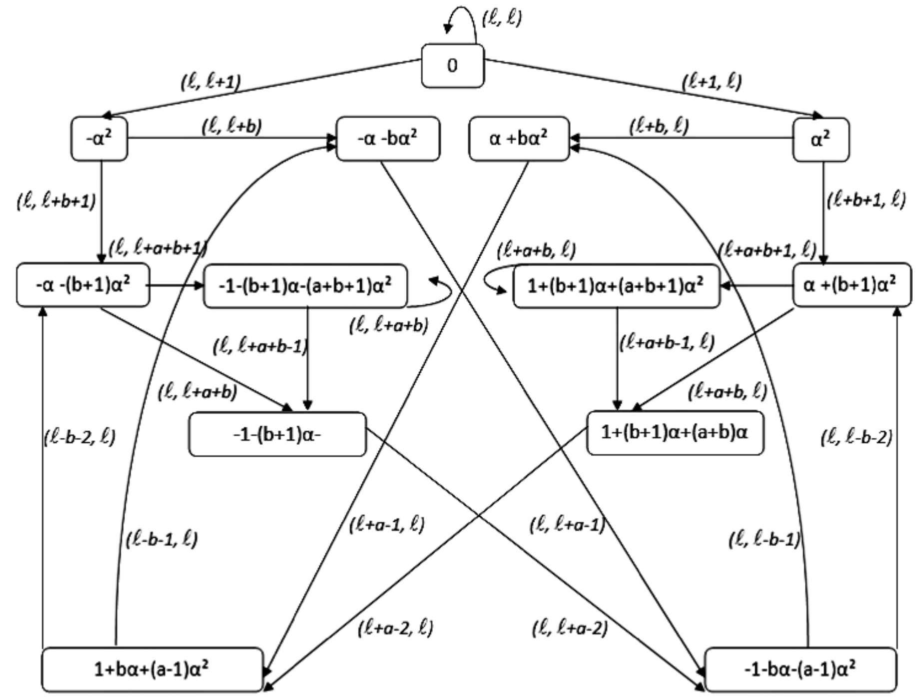

Fig. 5. Automaton $\mathcal{G}$.

Proof of Proposition 4.1. The direct implication is easy to see. Let us prove the converse. Let $x = \sum_{i=l}^{\infty} a_i\alpha^i$ and $y = \sum_{i=l}^{\infty} b_i\alpha^i$. Suppose that $x = y$, then $\alpha^{-k+2}x = \alpha^{-k+2}y$. Let us prove that the set $\{x(k) - y(k), k \geq 0\}$ is finite. Since $x(k) - y(k) = \alpha^{-k+2}(\sum_{i=0}^{k} a_i\alpha^i - \sum_{i=0}^{k} b_i\alpha^i) = \alpha^{-k+2}(\sum_{i=k+1}^{+\infty} b_i\alpha^i - \sum_{i=k+1}^{+\infty} a_i\alpha^i) = \sum_{j=3}^{+\infty}(b_{k+j-2} - a_{k+j-2})\alpha^j$, then $|x(k) - y(k)| \leq C$, where $C > 0$ is a constant.

Let $S_k = x(k) - y(k)$. Then S_k is an algebraic integer whose conjugates are $\widetilde{S}_k$ and $\overline{S}_k$, where $\widetilde{S}_k = \sum_{i=0}^{k}(a_i - b_i)\beta^{i-k+2}$ and $\overline{S}_k = \sum_{i=0}^{k}(a_i - b_i)\overline{\alpha}^{i-k+2}$.

We have $|\overline{S}_k| = |S_k| \leq C$, and

$$|\widetilde{S}_k| = \left|\sum_{i=l}^{k}(a_i - b_i)\beta^{i-k+2}\right| = |(a_0 - b_0)\beta^{-k+2} + \cdots + (a_k - b_k)\beta^2| \leq C\tfrac{\beta^2}{1-(1/\beta)},$$

where $C = 2 \cdot \max\{|a_i|, a_i \in \{0, 1, \ldots, a-1\}\} = 2(a-1)$. Then, there exists $M > 0$ such that S_k and all its conjugates are bounded by M, independently of k. Thus $\{S_k, k \geq 0\}$ is finite. □

As a consequence of this proposition, we have the following result.

Theorem 4.2. *Let $(a_i)_{i\geq l}$ and $(b_i)_{i\geq l}$ two distinct elements of $\mathcal{L}$, then $\sum_{i=l}^{\infty} a_i\alpha^i = \sum_{i=l}^{\infty} b_i\alpha^i$ if and only if the sequence $((a_i, b_i))_{i\geq l}$ is recognizable by the automaton $\mathcal{G}$.*

REMARK 4.3. The use of finite state automata to recognize points with two expansions is well known (see [34], [12], [22]). The difficulty is to find the states of these automata. And that is what we are going to do in the sequel.

Let us remind that $E_{a,b} = \{0, \pm\alpha^2, \pm(\alpha + b\alpha^2), \pm(\alpha + (b+1)\alpha^2), \pm(1 + b\alpha + (a-1)\alpha^2), \pm(1 + (b+1)\alpha + (a+b)\alpha^2), \pm(1 + (b+1)\alpha + (a+b+1)\alpha^2)\}$. To prove that $\bigcup_{(x,y)} J(x, y) \supset E_{a,b}$ we need the following result.

Proposition 4.4. *Let $F_{a,b} = \{S_k = n_k + p_k\alpha + q_k\alpha^2, k \geq 0,\ n_k,\ p_k,\ q_k \in \mathbb{Z}$ where S_k is a state of the automaton $\mathcal{G}\}$. Let $t = \max\{|n_k|, k \geq 0\}$. If $t = 1$, then $F_{a,b} = E_{a,b}$.*

For proving Proposition 4.4, we need the next lemma.

Lemma 4.5. *For all $k \geq l$, $|\widetilde{S}_k| = \left|\sum_{i=l}^{k}(a_i - b_i)\beta^{i-k+2}\right| < \beta^3$.*

Proof. Suppose, without loss of generality, that $\widetilde{S}_k = \sum_{i=l}^{k}(a_i - b_i)\beta^{i-k+2} > 0$. Then, $\widetilde{S}_k \in \mathbb{Z}[\beta] \cap \mathbb{R}^+$. Since β is a Pisot number, $\widetilde{S}_k = \sum_{i=-\infty}^{L} c_i\beta^i$, where $(c_i)_{i\leq L}$ is ultimately periodic (see [30]). Then, $\sum_{i=l}^{k} a_i\beta^{i-k+2} = \sum_{i=l}^{k} b_i\beta^{i-k+2} + \sum_{-\infty}^{L} c_i\beta^i$. Now, let us suppose that there exists $i \geq 3$ such that $c_i > 0$. Then $\sum_{i=l}^{k} a_i\beta^{i-k+2} \geq \beta^3$. Absurd, because $0a_2 \cdots a_l <_{lex} 10 \cdots 0$. Hence $L \leq 2$ and $\widetilde{S}_k < \beta^3$. □

Proof of Proposition 4.4. Let $S_k = n_k + p_k\alpha + q_k\alpha^2$ and $t = \max\{|n_k|, k \geq 0\}$. Let us suppose that $t = 1$. Then there exists and integer k such that $S_k = 1 + p\alpha + q\alpha^2$. Then, by (1), $S_{k+1} = \frac{1}{\alpha} + p + q\alpha^2$. Hence, $S_{k+1} = (p-b) + (q-a)\alpha + (d+1)\alpha^2$, where $d \in \Lambda = \{-a+1, ..., a-1\}$. Since $t = 1$, then $p \in \{b-1, b, b+1\}$.
Now we have to analyze all the possible values for p. Let us recall that $\beta^3 = a\beta^2 + b\beta + 1$.

CASE 1. $p = b$. In this case, $S_{k+1} = (q-a)\alpha + (d+1)\alpha^2$ and $S_{k+2} = (q-a) + (d+1)\alpha + e\alpha^2$, $e \in \Lambda$. Then, $q \in \{a-1, a, a+1\}$. We have $\tilde{S}_k = 1 + b\beta + q\beta^2$. By Lemma 4.5, we must have that $\tilde{S}_k < \beta^3$, hence $q = a-1$ because, otherwise, $\widetilde{S}_k \geq 1 + b\beta + q\beta^2 = \beta^3$. Hence we have the state $S_k = 1 + b\alpha + (a-1)\alpha^2$.

CASE 2. $p = b-1$. We have: $S_{k+1} = -1 + (q-a)\alpha + (d+1)\alpha^2$. Since $-S_{k+1} = 1 - (q-a)\alpha - (d+1)\alpha^2 \in F_{a,b}$, we obtain as before that $a - q \in \{b-1, b, b+1\}$. Let us show that these cases do not occur.

2.1. $q = a - b + 1$. In this case, $\tilde{S}_k = 1 + (b-1)\beta + (a-b+1)\beta^2 = \beta^3 - \beta + (1-b)\beta^2 \geq \beta^3 - \beta + 3\beta^2 > \beta^3$. Hence $\tilde{S}_k > \beta^3$.

2.2. $q = a - b$. We have $\tilde{S}_k = 1 + (b-1)\beta + (a-b)\beta^2 = \beta^3 - \beta - b\beta^2 \geq \beta^3 - \beta + 2\beta^2 > \beta^3$.

2.3. $q = a - b - 1$. In this case, $\tilde{S}_k = 1 + (b-1)\beta + (a-b-1)\beta^2 = \beta^3 - \beta + (-1-b)\beta^2 \geq \beta^3 - \beta + \beta^2 > \beta^3$. So we do not have the case when $p = b - 1$.

Case 3. $p = b + 1$. We have $S_{k+1} = 1 + (q-a) + (d+1)\alpha^2$ and $S_{k+2} = (q-a-b) + (d+1-a)\alpha + (e+1)\alpha^2$. Then, $q = a + b + r$, where $|r| \leq 1$. Hence,

$$\tilde{S}_k = 1 + p\beta + q\beta^2 = 1 + (b+1)\beta + (a+b+r)\beta^2,$$

with $|r| \leq 1$. We have to analyze all the possible cases for q.

3.1. $q = a+b$. In this case, $\tilde{S}_k = 1+(b+1)\beta+(a+b)\beta^2 = \beta^3+\beta+b\beta^2 \leq \beta^3+\beta-2\beta^2 < \beta^3$. So we have the state $S_k = 1 + (b+1)\alpha + (a+b)\alpha^2$.

3.2. $q = a + b + 1$. We have:

$$\tilde{S}_k = 1 + (b+1)\beta + (a+b+1)\beta^2 = \beta^3 + \beta + (b+1)\beta^2 \leq \beta^3 + \beta - \beta^2 < \beta^3.$$

Then we obtain the state $S_k = 1 + (b+1)\alpha + (a+b+1)\alpha^2$.

3.3. $q = a + b - 1$. In this case, $\tilde{S}_k = 1 + (b+1)\beta + (a+b-1)\beta^2$. Hence $S_{k+1} = 1 + (b-1)\alpha + (d+1)\alpha^2$, $d \in \Lambda$, which does not exist by Case 2. So, this case does not occur.

Let us now consider $S_k = n + p\alpha + q\alpha^2$ and suppose that $n = 0$. Then, $S_k = p\alpha + q\alpha^2$ and $A_{k+1} = p + q\alpha + d\alpha^2$. Then, $p \in \{-1, 0, 1\}$. Let us analyze all the possible cases, as we have done before.

Case 4. If $p = 0$ then $S_k = q\alpha^2$. So we obtain the states $S_k = 0$, if $q = 0$, and $S_k = \pm\alpha^2$, if $q = \pm 1$.

Case 5. If $p = 1$, then $S_k = \alpha + q\alpha^2$ and $S_{k+1} = 1 + q\alpha + d\alpha^2$. Hence, $S_{k+2} = (q-b) + (d-a)\alpha + (e+1)\alpha^2$. Thus, $q \in \{b-1, b, b+1\}$. Let us analyze the possible cases.

5.1. $q = b - 1$. This case does not occur, as seen in Case 2.

5.2. $q = b$. In this case we have the state $S_k = \alpha + b\alpha^2$.

5.3. $q = b + 1$. In this case we have the state $S_k = \alpha + (b+1)\alpha^2$. □

The next proposition tells us that the automaton could have other states depending on certain conditions.

Proposition 4.6. *Let t be the integer defined in* Proposition 4.4 *and suppose that* $1 < t \leq \frac{a-1}{a+b+1}$. *If $a + b \geq 3$ then $S_k = t + (t+tb)\alpha + (ta+tb+t)\alpha^2$ is a state of the automaton $\mathcal{G}$.*

Proof. This proof highly depends on the properties of the associated β−expansion and it must be divided into several cases. Let us remind that $d(1,\beta) = (a-1)(a+b-1)(a+b)^\infty$.

CASE 1. $t \geq 2$. Suppose that $S_k = t + p\alpha + q\alpha^2$. Then, by (1), $S_{k+1} = (p - tb) + (q - ta)\alpha + (f + t)\alpha^2$, where $|f| \leq a - 1$. Setting $p - tb = s$, where $s \in \{-t, \ldots, t\}$, thus $S_{k+1} = t + (q - ta)\alpha + (f + t)\alpha^2$. Then, $S_{k+2} = (q - ta - sb) + (f + t - sa)\alpha + (g + s)\alpha^2$, where $|g| \leq a - 1$. Setting $q = ta + sb + l$, for $l = -t, \ldots, t$ we obtain that $S_k = t + (tb + s)\alpha + (ta + sb + l)\alpha^2$. Let us show that $s = t$ whenever $a + b \geq 3$.

We have $\widetilde{S}_k = t + (tb + s)\beta + (ta + sb + l)\beta^2$. Since $t + tb\beta + ta\beta^2 = t\beta^3$, we obtain that $\widetilde{S}_k = t\beta^3 + (sb + l)\beta^2 + s\beta = \beta^3 + (t - 1)\beta^3 + (sb + l)\beta^2 + s\beta$.

Set $X = (t - 1)\beta^3 + (sb + l)\beta^2 + s\beta$. Using the fact that

$$\beta^2 = (a - 1)\beta + (a + b - 1) + (a + b) \sum_{i=1}^{\infty} 1/\beta^i,$$

we obtain

$$\begin{aligned} X/\beta &= (t - 1)\beta^2 + (sb + l)\beta + s \\ &= [(t - 1)(a - 1) + sb + l]\beta + [(t - 1)(a + b - 1) + s] + R, \end{aligned}$$

where $R = (t - 1)(a + b) \sum_{i=1}^{\infty} 1/\beta^i > 0$.

By Lemma 4.5 we must have $\widetilde{S}_k < \beta^3$. So, we need to show that $X/\beta \geq 0$. Let us do the first two cases. For all cases, the reader is referred to [25].
Let us suppose that $s < t$.

CASE 1.1. $-t \leq s \leq 0$. In this case,

$$\begin{aligned} X/\beta &= [(t - 1)(a - 1) + sb + l]\beta + [(t - 1)(a + b - 1) + s] + R \\ &\geq [l + (t - 1)(a - 1)]\beta + [(t - 1)(a + b - 1) + s] + R, \text{ since } sb \geq 0 \\ &\geq [(t - 1)(a - 1) - t]\beta + [(t - 1)(a + b - 1) - t] + R \end{aligned}$$

since $l \geq -t$ and $s \geq -t$.

CASE 1.1.1. $a + b - 1 \geq 2$. Since $t \geq 2$, $a \geq 3$ and $R > 0$ we obtain that $X/\beta \geq 0$ and then $X \geq 0$. So $\widetilde{S}_k = \beta^3 + X > \beta^3$, which is an absurd.

After analyzing all the possibles cases we conclude that $s = t$ and $l = t$.

CASE 2. $1 < m < t$. Let $S_k = m + p\alpha + q\alpha^2$, with $m < t$, then $S_{k+1} = (p - mb) + (q - ma)\alpha + (d + m)\alpha^2$. Hence, $S_{k+1} = s + (q - ma)\alpha + (d + m)\alpha^2$ and $S_{k+2} = (q - ma - sb) + (d + m - sa)\alpha + (g + s)\alpha^2$, $|s| \leq t$, $|q| \leq t$, $|l| \leq t$. So, $S_k = m + (mb + s)\alpha + (ma + sb + l)\alpha^2$.

REMARK. Since $ta + tb + t \in \mathcal{L}$, then it must satisfy the condition: $0 \leq ta + tb + t \leq a - 1$, that is, $t \leq \dfrac{a - 1}{a + b + 1}$.

Therefore, $S_k = t + (t + tb)\alpha + (ta + tb + t)\alpha^2$ is a state of the automaton $\mathcal{G}$. $\square$

Corollary 4.7. *The automaton has at least* $2(6 + 3(K - 1))$ *nonempty states, where* $K = \left\lfloor \frac{a-1}{a+b+1} \right\rfloor$. *The set of the states contains* $E_{a,b} \cup \{\pm t\alpha^2, \pm(t + t(b+1)\alpha^2), \pm(t + t(b+1)\alpha + t(a+b+1)\alpha^2)\}$, *where* $1 \leq t \leq K$.

The connection between the states of the automaton $\mathcal{G}$ and the neighbors of the Rauzy fractal $\mathcal{R}$ is contained in the proof of the next corollary.

Corollary 4.8. $\mathcal{R}$ *has at least* $6 + 2(K - 1)$ *neighbors of the form* $u + \mathcal{R}$, *where* $\pm u \in \{\pm\alpha, \pm(1 + b\alpha), \pm(1 + (b+1)\alpha)\} \cup \{\pm(t + t(b+1)\alpha\}$ *and* $2 \leq t \leq K$.

Proof. Let $S_k, k \geq 0$ be a state of the automaton $\mathcal{G}$. Assume that $S_0 = \alpha^2$. Then,

$$\alpha^2 + \sum_{i=3}^{\infty} \ell_i \alpha^i = \sum_{i=3}^{\infty} \ell'_i \alpha^i,$$

where $(\ell_i)_{i\geq 3}, (\ell'_i)_{i\geq 3} \in \mathcal{L}$. Hence,

$$\alpha + \sum_{i=3}^{\infty} \ell_i \alpha^{i-1} = \sum_{i=3}^{\infty} \ell'_i \alpha^{i-1}.$$

This implies that $\mathcal{R} \cap (\mathcal{R} + \alpha) \neq \emptyset$. Therefore, α is a neighbor of $\mathcal{R}$. The other neighbors are obtained in the same way. $\square$

From Corollary 4.8 we have the following theorem.

Theorem 4.9. *If* $2a + 3b + 4 \leq 0$ *then* $\mathcal{R}$ *is not homeomorphic to a topological disk.*

Proof. If $2a + 3b + 4 \leq 0$ then $3 \leq \frac{a-1}{a+b+1}$, that is, $K = \left\lfloor \frac{a-1}{a+b+1} \right\rfloor \geq 3$. Thus $\mathcal{R}$ has at least $6 + 2(K - 1) \geq 10$ neighbors. So $\mathcal{R}$ cannot be homeomorphic to a topological disk (see [13]). $\square$

Example 4.10. $\mathcal{R}_{8,-7}$ has at least 10 neighbors (see Fig. 6).

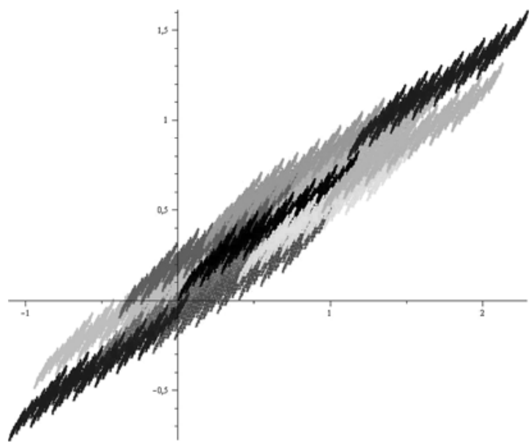

Fig. 6. $\mathcal{R}_{8,-7}$ and its neighbors.

5. Parametrization of the Boundary of $\mathcal{R}_{3,-2}$

In this section we will use the Automaton $\mathcal{G}$ built in the previous section with $a = 3$ and $b = -2$ to generate the boundary of $\mathcal{R} = \mathcal{R}_{3,-2}$. By Corollary 4.8, $\mathcal{R}_{3,-2}$ has at least 6 neighbors. Actually, we will show that $\mathcal{R}_{3,-2}$ has exactly 6 neighbors (see the Annex). We will also prove that the boundary of $\mathcal{R}$ is generated by two infinite countable sets of IFS.

Let $u \in \{\pm\alpha, \pm(1-\alpha), \pm(1-2\alpha)\}$ and denote by $\mathcal{R}_u = \mathcal{R} \cap (\mathcal{R} + u)$ the 6 curves which constitute the boundary of $\mathcal{R}$. The next proposition shows that each neighbor of $\mathcal{R}$ can be expressed by means of the other ones.

Proposition 5.1. *The following relations are valid:*
1. $\mathcal{R}_{1-\alpha} = \bigcup_{k=1}^{\infty} \ell_{k+1}\alpha^{k+1} + \alpha^k \mathcal{R}_{1-2\alpha}$*, where* $\ell_{k+1} \in \{0,1,2\}$*, for all* $k \geq 0$*.*
2. $\mathcal{R}_\alpha = \alpha\mathcal{R}_{1-2\alpha} \bigcup \bigcup_{k=1}^{\infty}(\ell_{k+1}\alpha^{k+2} + \alpha^{k+1}\mathcal{R}_{1-2\alpha})$*, where* $\ell_{k+1} \in \{0,1,2\}$*, for all* $k \geq 1$*.*

Proof. 1. Let $z \in \mathcal{R}_{1-\alpha}$. Then $z = 1 - \alpha + \sum_{i=2}^{+\infty} \ell_i\alpha^i = \sum_{i=2}^{\infty} \ell_i'\alpha^i$. So, by the Automaton $\mathcal{G}$, we have the associated paths in the automaton beginning in the initial state:

$$P_1 = (1,0)(-1,0)(\ell_2+1,\ell_2)(1,0)\cdots$$

or

$$P_2 = (1,0)(-1,0)(2,0)\underbrace{(1,0)(1,0)\cdots(1,0)}_{k-\text{times}}(\ell_{3+k},\ell_{3+k})(1,0)\cdots.$$

CASE 1.1. $z = 1 - \alpha + (\ell_2+1)\alpha^2 + \alpha^3 + \alpha^4 w = \ell_2\alpha^2 + \alpha^4 w'$, where $w, w' \in \mathbb{C}$. Hence, $z/\alpha - \ell_2\alpha = 1 - 2\alpha + 2\alpha^2 + \alpha^3 w = \alpha^3 w' \in \mathcal{R}_{1-2\alpha}$.

On the other hand, if $z \in \mathcal{R}_{1-2\alpha}$ then $z = 1 - 2\alpha + 2\alpha^2 + \alpha^3 w$, where $w \in \mathbb{C}$. Thus, $\alpha z + \ell_2\alpha^2 = \alpha + \ell_2\alpha^2 + 2\alpha^3 + \alpha^4 w = 1 + (\alpha - 2\alpha) + (\ell_2 - 3)\alpha^2 + \alpha^3 + \alpha^4 w = 1 - \alpha = (\ell_2+1)\alpha^2 + \alpha^3 + \alpha^4 w \in \mathcal{R}_{1-2\alpha}$, if $\ell_2 \geq 1$.

CASE 1.2.

$$z = 1 - \alpha + 2\alpha^2 + \underbrace{\alpha^3 + \cdots + \alpha^{k+2}}_{k-\text{times}} + \ell_{3+k}\alpha^{3+k} + \alpha^{4+k} + \alpha^{5+k}w_k = \ell_{3+k}\alpha^{3+k} + \alpha^{5+k}w_k',$$

where $w_k, w_k' \in \mathbb{C}$, for all $k \geq 0$. Hence, $z/\alpha^{k+2} = \alpha^{-k-2} - \alpha^{-k-1} + 2\alpha^{-k} + \alpha^{-k+1} + \cdots + 1 + \ell_{3+k}\alpha + \alpha^2 + \alpha^3 w_k = \ell_{3+k}\alpha + \alpha^3 w_k'$. Thus, by induction, we can show that

$$\frac{z}{\alpha^{k+2}} - \ell_{k+3}\alpha \in \mathcal{R}_{1-2\alpha}.$$

Therefore, $\mathcal{R}_{1-\alpha} = \bigcup_{k=1}^{\infty} \ell_{k+1}\alpha^{k+1} + \alpha^k\mathcal{R}_{1-2\alpha}$.

2. If $z \in \mathcal{R}_\alpha$ then $z = \alpha + \sum_{i=2} \ell_i\alpha^i = \sum_{i=2} \ell_i'\alpha^i$. Thus, $P_1 = (0,0)(1,0)(-2,0)(2,0)\cdots$ or $P_2 = (0,0)(1,0)(-1,0)\cdots$ are paths in the automaton beginning in the initial state.

CASE 2.1. $z = \alpha - 2\alpha^2 + 2\alpha^3 + \alpha^4 w_2 = \alpha^4 w_2'$. Hence, $z/\alpha = 1 - 2\alpha + 2\alpha^2 + \alpha^3 w_2 = \alpha^3 w_2' \in \mathcal{R}_{1-2\alpha}$.

CASE 2.2. Then $z = \alpha - \alpha^2 + \alpha^3 w_3 = \alpha^3 w_3'$. Hence, $z/\alpha = 1 - \alpha + \alpha^2 w_3 = \alpha^2 w_3' \in \mathcal{R}_{1-\alpha}$. (We are back in CASE 1.1).

Therefore, $\mathcal{R}_\alpha = \alpha\mathcal{R}_{1-2\alpha} \cup \alpha\mathcal{R}_{1-\alpha} = \alpha\mathcal{R}_{1-2\alpha} \bigcup \bigcup_{k=1}^{\infty} \ell_{k+1}\alpha^{k+2} + \alpha^{k+1}\mathcal{R}_{1-2\alpha}$. □

For all $z \in \mathbb{C}$, let us consider the iterated function system consisting of:

$f_0(z) = 3\alpha^4 - \alpha^5 + \alpha^3 z$,

$f_1(z) = -\alpha^2 + 2\alpha^3 + \alpha^2 z$,

$f_2(z) = -\alpha^3 + 4\alpha^4 - \alpha^5 + \alpha^3 z$,

$f_3(z) = \alpha^4 + 3\alpha^5 - \alpha^6 + \alpha^4 z$,

$f_4(z) = 2\alpha^4 + 3\alpha^6 - \alpha^7 + \alpha^5 z$,

$f_n(z) = 2\alpha^4 + (\sum_{j=1}^{n-4} \alpha^{4+j}) + 3\alpha^{n+2} - \alpha^{n+3} + \alpha^{n+1} z$, for all $n \geq 5$;

and

$g_0(z) = -\alpha^3 + 3\alpha^4 - \alpha^5 + \alpha^3 z$,

$g_1(z) = \alpha^4 + 2\alpha^5 - \alpha^6 + \alpha^4 z$,

$g_2(z) = 2\alpha^4 + 2\alpha^6 - \alpha^7 + \alpha^5 z$,

$g_n(z) = 2\alpha^4 + (\sum_{j=1}^{n-2} \alpha^{4+j}) + 2\alpha^{n+4} - \alpha^{n+5} + \alpha^{n+3} z$, for all $n \geq 3$.

Fig. 8 illustrates the behavior of this system. The next theorem shows that $\mathcal{R}_{1-2\alpha}$ is the infinite union of the images of itself by the applications defined above.

Theorem 5.2. $\mathcal{R}_{1-2\alpha} = \bigcup_{k=0}^{\infty} f_k(\mathcal{R}_{1-2\alpha}) \bigcup \bigcup_{k=0}^{\infty} g_k(\mathcal{R}_{1-2\alpha})$.

Proof. Let us recall that $\alpha^3 = 3\alpha^2 - 2\alpha + 1$. Since $\mathcal{R}_{1-2\alpha} = \mathcal{R} \cap \mathcal{R} + 1 - 2\alpha$, we have,

$f_0(\mathcal{R}_{1-2\alpha}) = f_0(\mathcal{R}) \cap f_0(\mathcal{R} + 1 - 2\alpha) = (3\alpha^4 - \alpha^5 + \alpha^3\mathcal{R}) \cap (\alpha^3 + \alpha^4 - \alpha^5 + \alpha^3\mathcal{R}) = (-\alpha^2 + 2\alpha^3 + \alpha^3\mathcal{R}) \cap (\alpha^3 + \alpha^4 - \alpha^5 + \alpha^3\mathcal{R}) = (1 - 2\alpha + 2\alpha^2 + \alpha^3\mathcal{R}) \cap (2\alpha^3 + \alpha^4 + \alpha^3\mathcal{R}) \subset \mathcal{R}_{1-2\alpha}$,

$f_1(\mathcal{R}_{1-2\alpha}) = f_1(\mathcal{R}) \cap f_1(\mathcal{R} + 1 - 2\alpha) = (-\alpha^2 + 2\alpha^3 + \alpha^2\mathcal{R}) \cap (\alpha^2\mathcal{R}) = (1 - 2\alpha + 2\alpha^2 + \alpha^3 + \alpha^2\mathcal{R}) \cap (\alpha^2\mathcal{R}) \subset \mathcal{R}_{1-2\alpha}$,

$f_2(\mathcal{R}_{1-2\alpha}) = f_2(\mathcal{R}) \cap f_2(\mathcal{R} + 1 - 2\alpha) = (-\alpha^3 + 4\alpha^4 - \alpha^5 + \alpha^3\mathcal{R}) \cap (2\alpha^4 - \alpha^5 + \alpha^3\mathcal{R}) = (1 - 2\alpha + 2\alpha^2 + \alpha^4 + \alpha^3\mathcal{R}) \cap (1 - 2\alpha + 2\alpha^2 + \alpha^3 + \alpha^3\mathcal{R}) \subset \mathcal{R}_{1-2\alpha}$,

$f_3(\mathcal{R}_{1-2\alpha}) = f_3(\mathcal{R}) \cap f_3(\mathcal{R} + 1 - 2\alpha) = (\alpha^4 + 3\alpha^5 - \alpha^6) \cap (2\alpha^4 + \alpha^5 - \alpha^6 + \alpha^4\mathcal{R}) = (-\alpha^2 + 2\alpha^3 + \alpha^4 - \alpha^5 + \alpha^4\mathcal{R}) \cap (1 - 2\alpha + 2\alpha^2 + \alpha^5 + \alpha^4\mathcal{R}) = (1 - 2\alpha + 2\alpha^2 + \alpha^3 + \alpha^4 + \alpha^4\mathcal{R}) \cap (1 - 2\alpha + 2\alpha^2 + \alpha^3 + \alpha^5 + \alpha^4\mathcal{R}) \subset \mathcal{R}_{1-2\alpha}$,

$f_4(\mathcal{R}_{1-2\alpha}) = f_4(\mathcal{R}) \cap f_4(\mathcal{R} + 1 - 2\alpha) = (2\alpha^4 + 3\alpha^6 - \alpha^7 + \alpha^5\mathcal{R}) \cap (2\alpha^4 + \alpha^5 + \alpha^6 - \alpha^7 + \alpha^5\mathcal{R}) = (\alpha^5\mathcal{R}) \cap (\alpha^5 - 2\alpha^6 + \alpha^5\mathcal{R}) \subset \mathcal{R} \cap (\mathcal{R} + 1 - 2\alpha) = \mathcal{R}_{1-2\alpha}$,

$f_n(\mathcal{R}_{1-2\alpha}) = f_n(\mathcal{R}) \cap f_n(\mathcal{R} + 1 - 2\alpha) = (2\alpha^4 + (\sum_{j=1}^{n-4} \alpha^{4+j}) + 2\alpha^{n+2} - \alpha^{n+3} + \alpha^{n+1}\mathcal{R}) \cap (2\alpha^4 + (\sum_{j=1}^{n-4} \alpha^{4+j}) + \alpha^{n+1} + \alpha^{n+2} - \alpha^{n+3} + \alpha^{n+1}\mathcal{R}) = (\alpha^{n+1}\mathcal{R}) \cap (\alpha^{n+1} - 2\alpha^{n+2} + \alpha^{n+1}\mathcal{R}) \subset \mathcal{R} \cap (\mathcal{R} + 1 - 2\alpha) = \mathcal{R}_{1-2\alpha}$, $\forall n \geq 5$.

Thus,

$$\bigcup_{k=0}^{\infty} f_k(\mathcal{R}_{1-2\alpha}) \subset \mathcal{R}_{1-2\alpha}.$$

On the other hand, let $z \in \mathcal{R}_{1-2\alpha}$. Using the automaton $\mathcal{G}$ we have the following paths beginning in the initial state:

1. $P_0 = (0,1)(0,-2)(1,1)(1,0)(1,0)\cdots$. Then, $z = \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 w_0 = 1 - 2\alpha + 2\alpha^2 + \alpha^3 + \alpha^6 w_0'$, where $w_0, w_0' \in \mathbb{C}$. Hence, $f_0^{-1}(z) = 1 - 2\alpha + 2\alpha^2 + \alpha^3 w_0 = \alpha^3 w_0' \in \mathcal{R}_{1-2\alpha}$, that is, $z \in f_1(\mathcal{R}_{1-2\alpha})$.

2. $P_1 = (0,1)(0,-2)(0,2)(0,1)(2,0)\cdots$. Then, $z = 2\alpha^4 + \alpha^5 w_0 = 1 - 2\alpha + 2\alpha^2 + \alpha^3 + \alpha^5 w_0'$, where $w_1, w_1' \in \mathbb{C}$. Hence, $f_1^{-1}(z) = 1 - 2\alpha + 2\alpha^2 + \alpha^3 w_1 = \alpha^3 w_1' \in \mathcal{R}_{1-2\alpha}$, that is, $z \in f_1(\mathcal{R}_{1-2\alpha})$.

3. $P_2 = (0,1)(0,-2)(0,2)(0,0)(2,1)(1,0)\cdots$. Then, $z = 2\alpha^4 + \alpha^5 + \alpha^6 w_2 = 1 - 2\alpha + 2\alpha^2 + \alpha^4 + \alpha^6 w_2'$. Thus, $f_2^{-1}(z) = 1 - 2\alpha + 2\alpha^2 + \alpha^3 w_2 = \alpha^3 w_2' \in \mathcal{R}_{1-2\alpha}$, that is, $z \in f_2(\mathcal{R}_{1-2\alpha})$.

4. $P_3 = (0,1)(0,-2)(0,2)(0,0)(2,0)(1,0)(1,0)\cdots$. Then, $z = 2\alpha^4 + \alpha^5 + \alpha^6 + \alpha^7 w_3 = 1 - 2\alpha + 2\alpha^2 + \alpha^7 w_3'$. Hence, $f_3^{-1}(z) = 1 - 2\alpha + 2\alpha^2 + \alpha^3 w_3 = \alpha^3 w_3' \in \mathcal{R}_{1-2\alpha}$, that is, $z \in f_3(\mathcal{R}_{1-2\alpha})$.

5. $P_4 = (0,1)(0,-2)(0,2)(0,0)(2,0)(1,0)(1,1)(1,0)\cdots$. Then, $z = 2\alpha^4 + \alpha^5 + \alpha^6 + \alpha^7 w_4 = 1 - 2\alpha + 2\alpha^2 + \alpha^6 + \alpha^7 w_4'$. Hence, $f_4^{-1}(z) = 1 - 2\alpha + 2\alpha^2 + \alpha^3 w_4 = \alpha^3 w_4' \in \mathcal{R}_{1-2\alpha}$, that is, $z \in f_4(\mathcal{R}_{1-2\alpha})$.

6. $P_5 = (0,1)(0,-2)(0,2)(0,0)(2,0)\underbrace{(1,0)(1,0)\cdots(1,0)}_{k-\text{times}}(1,1)(1,0)\cdots$. In this case, $z = 2\alpha^4 + \alpha^5 + \alpha^6 + \cdots + \alpha^{4+k} + \alpha^{5+k} + \alpha^{6+k} + \alpha^{7+k} w_{2+k} = 1 - 2\alpha + 2\alpha^2 + \alpha^{5+k} + \alpha^{7+k} w_{2+k}'$. Hence, $f_{2+k}^{-1}(z) = 1 - 2\alpha + 2\alpha^2 + \alpha^3 w_{2+k} = \alpha^3 w_{2+k}' \in \mathcal{R}_{1-2\alpha}$, for all $k \geq 2$, that is, $z \in f_{2+k}(\mathcal{R}_{1-2\alpha})$ for all $k \geq 2$.

In the same manner, we can do these calculations for the functions $g_i, i \geq 0$.

Therefore $\mathcal{R}_{1-2\alpha} = \bigcup_{k=0}^{\infty} f_k(\mathcal{R}_{1-2\alpha}) \bigcup \bigcup_{k=0}^{\infty} g_k(\mathcal{R}_{1-2\alpha})$. □

We have shown that $\mathcal{R}_{1-2\alpha} = \bigcup_{n\in\mathbb{N}} f_n(\mathcal{R}_{1-2\alpha}) \bigcup \bigcup_{n\in\mathbb{N}} g_n(\mathcal{R}_{1-2\alpha})$. Now, let $z \in \mathcal{R}_{1-2\alpha}$. Then, $z = f_{a_0}(z_0) = \psi_{a_0} \circ \psi_{a_1} \circ \cdots \circ \psi_{a_n}(z_n) = \lim_{n\to+\infty} \psi_{a_0} \circ \psi_{a_1} \circ \cdots \circ \psi_{a_n}(z_n)$, where $\psi_{a_i} = f_{a_i}$ or $\psi_{a_i} = g_{a_i}$, $a_i \in \mathbb{N}$, $z_n \in \mathcal{R}_{1-2\alpha}$ and z is fixed. Thus,

$$\mathcal{R}_{1-2\alpha} = \overline{\bigcup_{a_0,\cdots,a_n} \psi_{a_0} \circ \psi_{a_1} \circ \cdots \circ \psi_{a_n}(z)}.$$

Hence, using the Proposition 5.1 and the Theorem 5.2, we obtain the boundary of $\mathcal{R}_{3,-2}$ (see Fig. 11).

Parametrization of $\mathcal{R}_{1-2\alpha}$. The next lemma shows points of the fractal $\mathcal{R}$ that can be expressed in three different ways, i.e., points with three $\alpha-$representations. Consequently these points lie in the intersection of three neighbors of the fractal (see Remark 3.5). These points are shown in Figure 7.

Lemma 5.3. *The following properties are satisfied.*

1. $\mathcal{R}_\alpha \cap \mathcal{R}_{1-\alpha} = \alpha + \frac{\alpha^3+\alpha^4+\alpha^5}{1-\alpha^6}$
2. $\mathcal{R}_{1-\alpha} \cap \mathcal{R}_{1-2\alpha} = \frac{\alpha^4+\alpha^5+\alpha^6}{1-\alpha^6}$.
3. $\mathcal{R}_{1-2\alpha} \cap \mathcal{R}_{-\alpha} = \frac{\alpha^3+\alpha^4+\alpha^5}{1-\alpha^6}$.
4. $\mathcal{R}_{-\alpha} \cap \mathcal{R}_{-1+\alpha} = -1 + \alpha + \frac{\alpha^4+\alpha^5+\alpha^6}{1-\alpha^6}$.
5. $\mathcal{R}_{-1+\alpha} \cap \mathcal{R}_{-1+2\alpha} = -1 + 2\alpha + \frac{\alpha^3+\alpha^4+\alpha^5}{1-\alpha^6}$.
6. $\mathcal{R}_{-1+2\alpha} \cap \mathcal{R}_\alpha = -1 + 2\alpha + \frac{\alpha^4+\alpha^5+\alpha^6}{1-\alpha^6}$.

Proof. 1. If $w \in \mathcal{R}_\alpha \cap \mathcal{R}_{1-\alpha}$ then $w = \alpha + \sum_{i=2}^{+\infty} \ell_i \alpha^i = 1-\alpha+\sum_{i=2}^{+\infty} \ell'_i \alpha^i$, where $(\ell_i)_{i\geq 2}, (\ell'_i)_{i\geq 2} \in \mathcal{L}$. Hence, using the automaton $\mathcal{G}$, we obtain that

$$\begin{aligned} w &= 1-\alpha+2\alpha^2+\alpha^3+\textstyle\sum_{i=1}^{\infty}(\alpha^{6i+1}+\alpha^{6i+2}+\alpha^{6i+3}) = \alpha^2 + \sum_{i=1}^{\infty}(\alpha^{6i-1}+\alpha^{6i}+\alpha^{6i+1}) \\ &= \alpha + \textstyle\sum_{i=1}^{\infty}(\alpha^{6i-3}+\alpha^{6i-2}+\alpha^{6i-1}) = \alpha + \frac{\alpha^3+\alpha^4+\alpha^5}{1-\alpha^6}. \end{aligned}$$

2. If $x \in \mathcal{R}_{1-\alpha} \cap \mathcal{R}_{1-2\alpha}$ then $x = 1-\alpha+\sum_{i=2}^{+\infty} \ell_i \alpha^i = 1-2\alpha+\sum_{i=2}^{+\infty} \ell'_i \alpha^i$, where $(\ell_i)_{i\geq 2}, (\ell'_i)_{i\geq 2} \in \mathcal{L}$. Using the Automaton we obtain that

$$\begin{aligned} x &= 1-\alpha+\textstyle\sum_{i=1}^{\infty}(\alpha^{6i-4}+\alpha^{6i-3}+\alpha^{6i-2}) = 1-2\alpha+2\alpha^2+\sum_{i=1}^{\infty}(\alpha^{6i-4}+\alpha^{6i-3}+\alpha^{6i-2}) \\ &= \textstyle\sum_{i=1}^{\infty}(\alpha^{6i-2}+\alpha^{6i-1}+\alpha^{6i}) = \frac{\alpha^4+\alpha^5+\alpha^6}{1-\alpha^6}. \end{aligned}$$

The other relations come from the fact that: $\mathcal{R}_{1-2\alpha} \cap \mathcal{R}_{-\alpha} = \mathcal{R}_\alpha \cap \mathcal{R}_{1-\alpha} - \alpha$, $\mathcal{R}_{-\alpha} \cap \mathcal{R}_{-1+\alpha} = \mathcal{R}_{1-\alpha} \cap \mathcal{R}_{1-2\alpha} - 1 + \alpha$, $\mathcal{R}_{-1+\alpha} \cap \mathcal{R}_{-1+2\alpha} = \mathcal{R}_\alpha \cap \mathcal{R}_{1-\alpha} - 1 + \alpha$, and $\mathcal{R}_{-1+2\alpha} \cap \mathcal{R}_\alpha = \mathcal{R}_{1-\alpha} \cap \mathcal{R}_{1-2\alpha} - 1 + 2\alpha$. □

Proposition 5.4. *For all $i, l \in \mathbb{N}$,*

1. *$f_i(\mathcal{R}_{1-2\alpha}) \cap f_l(\mathcal{R}_{1-2\alpha}) \neq \emptyset$ if, and only if, $0 \leq |i-l| \leq 1$. In particular, $f_k(\mathcal{R}_{1-2\alpha}) \cap f_{k+1}(\mathcal{R}_{1-2\alpha}) = \{f_k(z_0)\} = \{f_{k+1}(y_0)\}$, where $z_0 = \frac{\alpha^3+\alpha^4+\alpha^5}{1+\alpha^6}$ and $y_0 = \frac{\alpha^4+\alpha^5+\alpha^6}{1-\alpha^6}$;*

2. *$g_i(\mathcal{R}_{1-2\alpha}) \cap g_l(\mathcal{R}_{1-2\alpha}) \neq \emptyset$ if, and only if, $0 \leq |i-l| \leq 1$. In particular, $g_k(\mathcal{R}_{1-2\alpha}) \cap g_{k+1}(\mathcal{R}_{1-2\alpha}) = \{g_k(y_0)\} = \{g_{k+1}(z_0)\}$, where $z_0 = \frac{\alpha^3+\alpha^4+\alpha^5}{1+\alpha^6}$ and $y_0 = \frac{\alpha^4+\alpha^5+\alpha^6}{1-\alpha^6}$;*

3. *$f_i(\mathcal{R}_{1-2\alpha}) \cap g_l(\mathcal{R}_{1-2\alpha}) = \emptyset$, for all $i, l \in \mathbb{N}$.*

Proof. Let us prove the item 1.

Case $0 \leq |i-l| \leq 1$.

Let us suppose that $w \in f_k(\mathcal{R}_{1-2\alpha}) \cap f_{k+1}(\mathcal{R}_{1-2\alpha})$. Then there exists $y, z \in \mathcal{R}_{1-2\alpha}$ such that

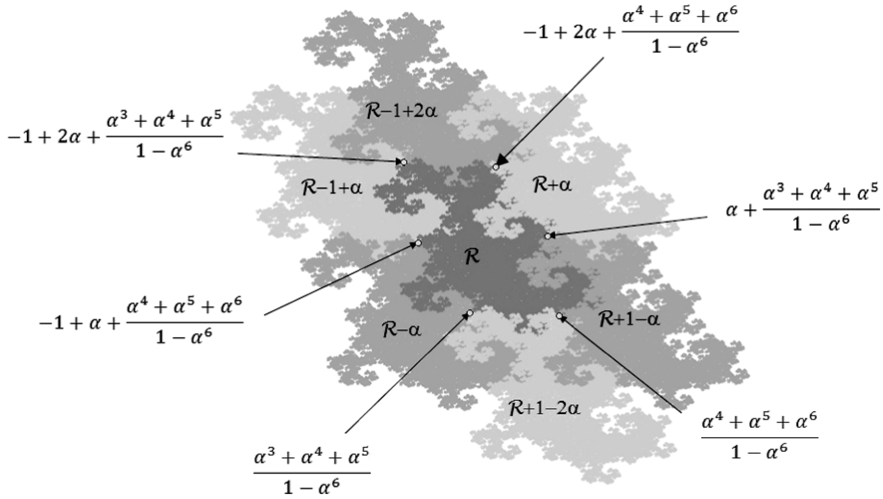

Fig. 7. Points with three α-representations.

$y = -\alpha + \alpha^2 + \alpha z \in \mathcal{R}_{-\alpha} \cap \mathcal{R}_{1-2\alpha}$. Hence, $y = \{z_0\}$ and $z = \{y_0\}$. Therefore, $f_k(\mathcal{R}_{1-2\alpha}) \cap f_{k+1}(\mathcal{R}_{1-2\alpha}) = \{f_k(z_0)\} = \{f_{k+1}(y_0)\}$.

In the same way, we can show that $f_0(\mathcal{R}_{1-2\alpha}) \cap f_1(\mathcal{R}_{1-2\alpha}) = \{f_0(z_0)\} = \{f_1(y_0)\}$, $f_1(\mathcal{R}_{1-2\alpha}) \cap f_2(\mathcal{R}_{1-2\alpha}) = \{f_1(z_0)\} = \{f_2(y_0)\}$, and $f_2(\mathcal{R}_{1-2\alpha}) \cap f_3(\mathcal{R}_{1-2\alpha}) = \{f_2(z_0)\} = \{f_3(y_0)\}$.

CASE $|i - l| > 1$.

Suppose that $l > i$ and that $f_i(\mathcal{R}_{1-2\alpha}) \cap f_l(\mathcal{R}_{1-2\alpha}) \neq \emptyset$. Then there exists $y, z \in \mathcal{R}_{1-2\alpha}$ such that

$$\sum_{j=1}^{i-4} \alpha^{4+j} + 3\alpha^{i+2} - \alpha^{i+3} + \alpha^{i+1} y = \sum_{j=1}^{l-4} \alpha^{4+j} + 3\alpha^{l+2} - \alpha^{l+3} + \alpha^{l+1} z. \tag{2}$$

Since $y, z \in \mathcal{R}_{1-2\alpha}$, they can be expressed as $y = 1 - 2\alpha + 2\alpha^2 + \alpha^3 \bar{y}$ and $z = 1 - 2\alpha + 2\alpha^2 + \alpha^3 \bar{z}$, where $\bar{y}, \bar{z} \in \mathbb{C}$. Replacing this in the equation (2) we obtain that

$$\bar{y} = 1 + \alpha + \alpha^2 + \cdots + \alpha^{l-i-1} + \alpha^{l-i}(\bar{z}). \tag{3}$$

Thus,

$$\underbrace{(1,0)(1,0)(1,0)\ldots(1,0)}_{(l-i-1)\text{ times}}\ldots$$

is the associated path in the automaton beginning in the initial state that represents the point in (3). Absurd, because there is no such a path in the automaton.

Therefore, $f_i(\mathcal{R}_{1-2\alpha}) \cap f_l(\mathcal{R}_{1-2\alpha}) = \emptyset$.

Using the same reasoning we can prove the items 2. and 3. □

Now we show a geometric way for constructing $\mathcal{R}_{1-2\alpha}$. Fig. 9 illustrates this procedure. Let z_0 and y_0 be two end points of $\mathcal{R}_{1-2\alpha}$ as in Proposition 5.4. Let us consider the sequence of functions $\varphi_n : [0,1] \to \mathbb{C}, n \geq 1$, where:

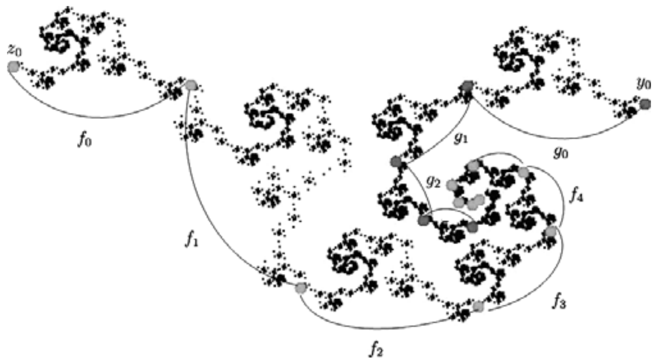

Fig. 8. $\mathcal{R}_{1-2\alpha}$.

$\varphi_1([0,1])$ is the polygonal line consisting of segments of the form $[f_k(y_0),\ f_{k+1}(y_0)]$, for $k \in \mathbb{N}$, and the segments $[g_k(z_0),\ g_{k+1}(z_0)]$, for $k \in \mathbb{N}$. Let us remark that, by Proposition 5.4, they could be joint in a continuous way (see Fig. 9).

$\varphi_2([0,1])$ is the polygonal line consisting of all of the segments $[f_i \circ f_j(x), f_i \circ f_{j+1}(x)]$, $[f_i \circ g_j(x), f_i \circ g_{j+1}(x)]$, where $x \in \{z_0, y_0\}$, and $i, j \in \mathbb{N}$ (see Fig. 9). Notice that, to pass from $\varphi_1([0,1])$ to $\varphi_2([0,1])$ we subdivide each interval $[f_k(y_0), f_{k+1}(y_0)]$ (respectively $[g_k(z_0), g_{k+1}(z_0)])$, $k \in \mathbb{N}$, in infinitely many intervals in the following way:

If $k = 0$, we join the intervals in this order: $[f_0 \circ f_0(y_0), f_0 \circ f_1(y_0)]$, $[f_0 \circ f_1(y_0), f_0 \circ f_2(y_0)], \cdots, [f_0 \circ f_{k-1}(y_0), f_0 \circ f_k(y_0)], [f_0 \circ g_k(z_0), f_0 \circ g_{k-1}(z_0)], \cdots, [f_0 \circ g_1(z_0), f_0 \circ g_0(z_0)]$.

If $k = 1$, we join the intervals in this order: $[f_1 \circ f_1(y_0), f_1 \circ f_2(y_0)]$, $[f_1 \circ f_2(y_0), f_1 \circ f_3(y_0)], \cdots, [f_1 \circ f_{k-1}(y_0), f_1 \circ f_k(y_0)], [f_1 \circ g_k(z_0), f_1 \circ g_{k-1}(z_0)], \cdots, [f_1 \circ g_1(z_0), f_1 \circ g_0(z_0)]$, and so on.

Once $\varphi_n([0,1])$ has been constructed, $\varphi_n([0,1]) = \bigcup [f_i \circ \psi_{j_1} \circ \cdots \circ \psi_{j_n}(x), f_i \circ \psi_{j_{i+1}} \circ \cdots \circ \psi_{j_{n+1}}(x)]$, where $x \in \{z_0, y_0\}$, and $\psi_j = f_j$ or $\psi_j = g_j$, for $i, j \in \mathbb{N}$ (see Figure 10 for clarity). We have the following result.

Proposition 5.5. *Let $(\varphi_n)_{n\geq 0}$ be the sequence of functions where $\varphi_n : [0,1] \to \mathbb{C}$ are defined as above. Then $(\varphi_n([0,1]))_{n\geq 0}$ converges to a compact set in the Hausdorff distance.*

Proof. The Haudorff distance between two sets X and Y is defined by

$$d_H(X, Y) = \max\left\{\max_{x\in X} \inf_{y\in Y} |x-y|, \max_{y\in Y} \inf_{x\in X} |x-y|\right\}.$$

Take $y \in \varphi_{n+1}([0,1]), n \in \mathbb{N}$. Then, $y = \psi_{a_0} \circ \psi_{a_1} \circ \cdots \circ \psi_{a_{n+1}}(z)$, where $z \in \{z_0, y_0\}$, and $\psi_{a_i} = f_{a_i}$ or $\psi_{a_i} = g_{a_i}$, $a_i \in \mathbb{N}$. Thus,

$$\inf_{x\in\varphi_n([0,1])} |x-y| \leq |\psi_{a_0} \circ \psi_{a_1} \circ \cdots \circ \psi_{a_n}(z) - \psi_{a_0} \circ \psi_{a_1} \circ \cdots \circ \psi_{a_{n+1}}(z)| \leq C|\alpha|^n,$$

where $C = \max\{|z| : z \in \mathcal{R}_{1-2\alpha}\}$. Hence,

$$\max_{y \in \varphi_{n+1}([0,1])} \inf_{x \in \varphi_n([0,1])} |x - y| \leq C|\alpha|^n.$$

We can infer the same estimate for $\max_{x \in \varphi_n([0,1])} \inf_{y \in \varphi_{n+1}([0,1])} |x - y|$. Therefore,

$$d_H(\varphi_n([0,1]), \varphi_{n+1}([0,1])) \leq |\alpha|^n \dot{C}.$$

Since $|\alpha| < 1$, then $(\varphi_n([0,1]))_{n \geq 0}$ converges to a compact set $\mathcal{K}$. It is not difficult to see that $\mathcal{K} = \mathcal{R}_{1-2\alpha}$. □

Notice that with this method we can parametrize the whole boundary of $\mathcal{R}_{3,-2}$, once each neighbor is expressed by means of the other ones.

Concluding Remarks. The methods used in [7] maybe can be applied, with some natural modifications to comprise our case of infinite iterated function systems, to obtain further topological properties of the boundary of $\mathcal{R}_{3,-2}$, or more generally, of the boundaries of $\mathcal{R}_{a,b}$, such as its Hausdorff dimension and the disk-like property.

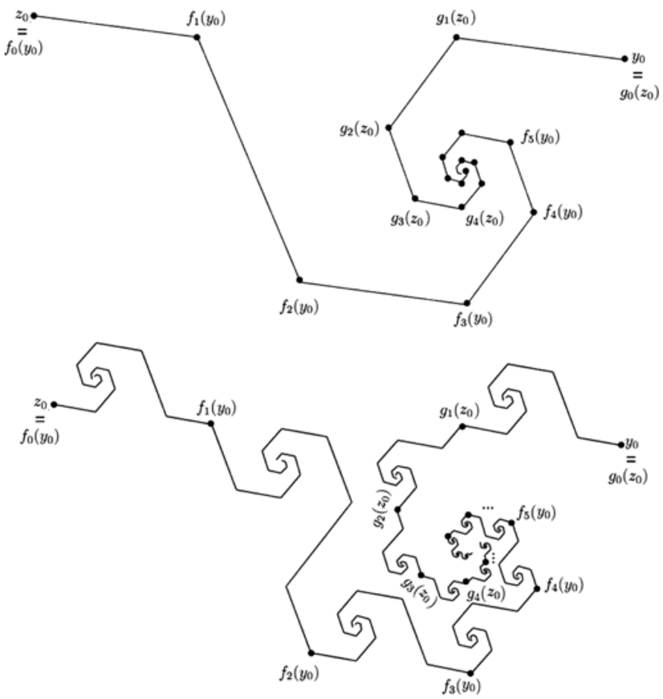

Fig. 9. Approximating $\mathcal{R}_{1-2\alpha}$ by $\varphi_1([0,1])$ and $\varphi_2([0,1])$.

6. Annex

In this section we will give an explicit proof that $\mathcal{R} = \mathcal{R}_{3,-2}$ has no more than 6 neighbors.

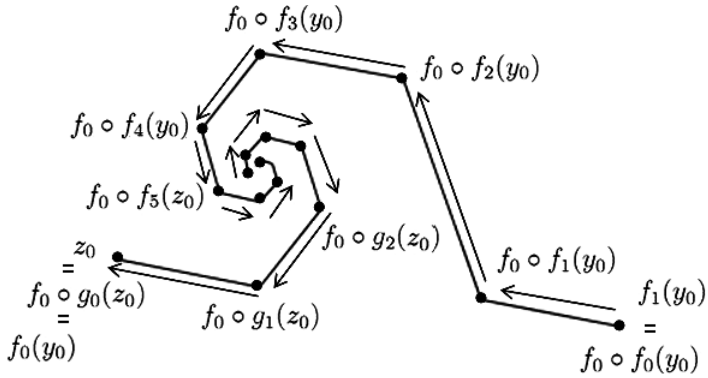

Fig. 10. Zoom of the interval $[f_0(y_0), f_1(y_0)]$ in $\varphi_2([0, 1])$.

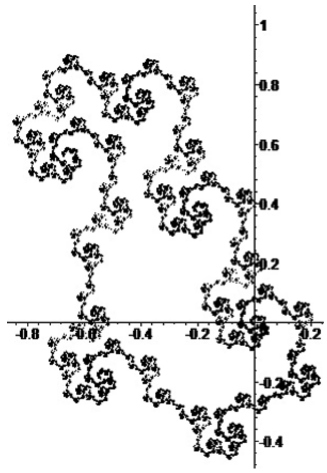

Fig. 11. Boundary of $\mathcal{R}_{3,-2}$.

Proposition 6.1. *The Rauzy fractal $\mathcal{R} = \mathcal{R}_{3,-2}$ has exactly 6 neighbors of the form $\mathcal{R} + u$, where $u \in \{\pm\alpha, \pm 1 - 2\alpha, \pm 1 - \alpha\}$, i.e.,*

$$\forall u \in \mathbb{Z} + \mathbb{Z}\alpha \backslash \{0\},\ \mathcal{R} \cap \mathcal{R} + u \neq \emptyset \Longleftrightarrow u \in \{\pm\alpha, \pm 1 - 2\alpha, \pm 1 - \alpha\}.$$

Proof. Let us suppose that $\mathcal{R} \cap (\mathcal{R} + p + q\alpha) \neq \emptyset$, where $p, q \in \mathbb{Z}$. Then there exists $z \in \mathcal{R}$ such that

$$z = \sum_{i=2}^{=\infty} \ell_i \alpha^i = p + q\alpha + \sum_{i=2}^{=\infty} \ell'_i \alpha^i, \text{ where } (\ell_i)_{i\geq 2}, (\ell'_i)_{i\geq 2} \in \mathcal{L}.$$

On the other hand, we can rearrange the terms as:

$$\sum_{i=2}^{+\infty} (\ell_i - \ell'_i)\alpha^i = \sum_{i=0}^{+\infty} \gamma_{4i} \alpha^{4i+2},$$

where $\gamma_{4i} = (\ell_{2+4i} - \ell'_{2+4i}) + \cdots + (\ell_{5+4i} - \ell'_{5+4i})\alpha^3$.

Hence,

$$|p + q\alpha|^2 \leq \left(\frac{k|\alpha|^2}{1 - |\alpha|^4} \right)^2,$$

where $k = \max\{\sum_{i=0}^{3} |\ell_i - \ell_i'||\alpha|^i,\ \ell_i, \ell_i' \in \{0,1,2\}\}$. We can show that $\left(\frac{k|\alpha|^2}{1-|\alpha|^4}\right)^2 \leq 2.4256$.

Now, let us set $\alpha = c + di$, where $c \approx 0,3376$, and $d \approx 0,5622$. Hence, $|p + q\alpha|^2 = (p + qc)^2 + q^2d^2$. Therefore, $|q| \leq 2$, because if $|q| \geq 3$ we would have $q^2d^2 \geq 2.8446 > 2.4256$. Then, we have to analyze the cases where $q \in \{0, \pm 1, \pm 2\}$:

CASE 1. If $q = -2$ then $p = 1$. Indeed, $|p+q\alpha|^2 = (p-2c)^2+4d^2 \leq 2.4256 \Longrightarrow (p-2c)^2 \leq 2.4256 - 1.2642 = 1.1613$. In this case, the only possibility is $p = 1$. Then we have the neighbor $\mathcal{R} + 1 - 2\alpha$.

CASE 2. If $q = -1$ then $p \in \{0, 1\}$. Indeed, $|p+q\alpha|^2 = (p-c)^2+d^2 \leq 2.4256 \Longrightarrow (p-c)^2 \leq 2.4256 - 0.3160 = 2.1096$. In this case, the possible values for p are $\{0, 1\}$ and then we have the neighbors $\mathcal{R} - \alpha$, $\mathcal{R} + 1 - \alpha$, and $\mathcal{R} + 2 - 2\alpha$.

CASE 3. If $q = 0$ then $p \in \{-1, 0, 1\}$. Indeed, $|p + q\alpha|^2 = p^2 \leq 2.4256$ and the possible values for p are $\{0, 1\}$ and then we have the neighbors $\mathcal{R} - \alpha$, $\mathcal{R} + 1 - \alpha$, and $\mathcal{R} + 2 - 2\alpha$.

CASE 4. If $q = 1$ then $p \in \{-1, 0, 1\}$. Indeed, $|p + q\alpha|^2 = (p + c)^2 + d^2 \leq 2.4256 \Longrightarrow (p + c)^2 \leq 2.4256 - 0.3160 = 2.1096$. In this case, the possible values for p are $\{-1, 0, 1\}$ and then we have the neighbors $\mathcal{R} - 1$ and $\mathcal{R} + 1$.

CASE 5. If $q = 2$ then $p \in \{-1, 0, 1\}$. Indeed, $|p + q\alpha|^2 = (p + 2c)^2 + 4d^2 \leq 2.4256 \Longrightarrow (p + 2c)^2 \leq 2.4256 - 1.2542 = 1.1613$, and the possible values for p are $\{-1, 0, 1\}$ and then we have the neighbors $\mathcal{R} - 1 + 2\alpha$,$\mathcal{R} + 2$, and $\mathcal{R} + 1 + 2\alpha$.

Thus, we have found the neighbors: $\mathcal{R} \pm \alpha$, $\mathcal{R} \pm (1 - \alpha)$, $\mathcal{R} \pm (1 - 2\alpha)$, $\mathcal{R} \pm 1$, $\mathcal{R} \pm (1 + \alpha)$, $\mathcal{R} + 2$, $\mathcal{R} + 1 + 2\alpha$. Let us see how to exclude the last six neighbors.

CASE 1. Suppose that $\mathcal{R} \cap \mathcal{R} + 1 + 2\alpha \neq \emptyset$. Then, $1 + 2\alpha = \sum_{i=2}^{+\infty}(\ell_i - \ell_i')\alpha^i$, where $(\ell_i)_{i\geq 2}, (\ell_i')_{i\geq 2} \in \mathcal{L}$. Hence,

$$|1 + 2\alpha + (\ell_2' - \ell_2)\alpha^2| = |\sum_{i=3}^{+\infty}(\ell_i - \ell_i')\alpha^i| < |\alpha| \cdot 2.4256 \leq 1.59. \tag{4}$$

Let us recall that $|\alpha| \approx 0.6558$. On the other hand we have,

$$|1+2\alpha+(\ell_2'-\ell_2)\alpha^2| \in \{|1+2\alpha|, |1+2\alpha+\alpha^2|, |1+2\alpha-\alpha^2|, |1+2\alpha+2\alpha^2|, |1+2\alpha-2\alpha^2|\} \approx \Gamma_1, \tag{5}$$

where $\Gamma_1 = \{2.0177, 2.1054, 2.0198, 2.2725, 2.1114\}$. By (4) and (5) we have a contradiction.

CASE 2. Suppose that $\mathcal{R} \cap \mathcal{R} + 2 \neq \emptyset$. Then, $2 = \sum_{i=2}^{+\infty}(\ell_i - \ell_i')\alpha^i$, where $(\ell_i)_{i\geq 2}, (\ell_i')_{i\geq 2} \in \mathcal{L}$. Hence,

$$|2 + (\ell_2' - \ell_2)\alpha^2| = |\sum_{i=3}^{+\infty}(\ell_i - \ell_i')\alpha^i| < |\alpha| \cdot 2.4256 \leq 1.59. \tag{6}$$

On the other hand,

$$|2 + (\ell_2' - \ell_2)\alpha^2| \in \{2, |2 + \alpha^2|, |2 - \alpha^2|, |2 + 2\alpha^2|, |2 - 2\alpha^2|\} \approx \Gamma_2, \tag{7}$$

where $\Gamma_2 = \{1.8375, 2.2346, 1.7671, 2.5213\}$. By (6) and (7) we have a contradiction.

CASE 3. Suppose that $\mathcal{R} \cap \mathcal{R} + 1 \neq \emptyset$. Then, $1 = \sum_{i=2}^{+\infty}(\ell_i - \ell_i')\alpha^i$, where $(\ell_i)_{i\geq 2}, (\ell_i')_{i\geq 2} \in \mathcal{L}$. Hence,

$$|1 + (\ell_2' - \ell_2)\alpha^2| = |\sum_{i=3}^{+\infty}(\ell_i - \ell_i')\alpha^i| < |\alpha| \cdot 2.4256 \leq 1.59. \tag{8}$$

On the other hand,

$$|1 + (\ell_2' - \ell_2)\alpha^2| \in \{1, |1 + \alpha^2|, |1 - \alpha^2|, |1 + 2\alpha^2|, |1 - 2\alpha^2|\} \approx \Gamma_3, \tag{9}$$

where $\Gamma_3 = \{1, 0.8835, 1.2606, 0.9651, 1.5964\}$. By (8) and (9) we have a contradiction.

CASE 4. Suppose that $\mathcal{R} \cap \mathcal{R} + 1 + \alpha \neq \emptyset$. Then, $1 + \alpha = \sum_{i=2}^{+\infty}(\ell_i - \ell_i')\alpha^i$, where $(\ell_i)_{i\geq 2}, (\ell_i')_{i\geq 2} \in \mathcal{L}$. Hence,

$$|1 + \alpha + (\ell_2' - \ell_2)\alpha^2| = |\sum_{i=3}^{+\infty}(\ell_i - \ell_i')\alpha^i| < |\alpha| \cdot 2.4256 \leq 1.59. \tag{10}$$

On the other hand,

$$|1 + \alpha + (\ell_2' - \ell_2)\alpha^2| \in \{|1 + \alpha|, |1 + \alpha + \alpha^2|, |1 + \alpha - \alpha^2|, |1\alpha + 2\alpha^2|, |1 + \alpha - 2\alpha^2|\} \approx \Gamma_4, \tag{11}$$

where $\Gamma_4 = \{1.4510, 1.4753, 1.5505, 1.6179, 1.7530\}$. By (10) and (11) we have a contradiction. Therefore, $\mathcal{R}_{3,-2}$ has only 6 neighbors. □

ACKNOWLEDGEMENTS. I deeply thank Ali Messaoudi for all the help during the writing of the manuscript. I would also like to thank the anonymous referee for carefully reading the manuscript and whose suggestions helped to improve a lot the readability of the paper.

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Department of Mathematics
São Paulo State University
UNESP
Rua Cristóvão Colombo, 2265
Jardim Nazareth, 15054-000
São José do Rio Preto, SP
Brazil
e-mail: pavani@ibilce.unesp.br